

F-theory Models without Section on Calabi-Yau 4-folds

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Abstract

We investigate gauge theories and matter contents in F-theory compactifications on families of genus-one fibered Calabi–Yau 4-folds lacking a global section. To construct families of genus-one fibered Calabi–Yau 4-folds that lack a global section, we consider two constructions: hypersurfaces in a product of projective spaces, and double covers of a product of projective spaces. We consider specific forms of defining equations for these genus-one fibrations, so that genus-one fibers possess complex multiplications of specific orders. These symmetries enable a detailed analysis of gauge theories. E_6 and E_7 gauge groups arise in some models. Discriminant components intersect with one another in the constructed models, and therefore, discriminant components contain matter curves. We deduce potential matter spectra and Yukawa couplings.

We also determine the Jacobian fibrations of the constructed Calabi–Yau genus-one fibrations. We compute the Mordell–Weil groups of some specific Jacobians. We obtain \mathbb{Z}_3 and \mathbb{Z}_2 groups.

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1 Introduction

F-theory [1–3] is a framework that extends the type IIB superstring theory to a nonperturbative regime, and the compactification geometries for F-theory are Calabi–Yau manifolds with a torus fibration. In the F-theory approach, the modular parameter of a genus-one curve, as a fiber of a torus fibration, is identified with the axio-dilaton; this formulation enables the axio-dilaton to have $SL_2(\mathbb{Z})$ monodromy.

A Calabi–Yau manifold with a torus fibration may or may not admit a global section. F-theory models on Calabi–Yau manifolds with a global section have been studied previously, for example, in [4–19]. In recent years, there has been an increasing interest in F-theory models on Calabi–Yau genus-one fibrations without a global section¹. Initiated in [22, 23], F-theory compactifications lacking a global section have been discussed in recent studies. See also, for example, [24–34] for recent advances in F-theory models that lack a global section. It was argued in [23] that, by considering the Jacobian fibrations, the F-theory models on Calabi–Yau genus-one fibrations without a global section can be related to the geometry of Calabi–Yau elliptic fibrations with a section.

In this note, we construct genus-one fibered Calabi–Yau 4-folds without a global section, and we use these spaces as compactification geometries for F-theory to investigate F-theory models without a section. We consider two constructions: hypersurfaces in a product of projective spaces, and double covers of a product of projective spaces, to construct genus-one fibered Calabi–Yau 4-folds without a rational section. In these constructions, we only consider Calabi–Yau 4-folds whose structures are complex enough that discriminant components intersect with one another. Therefore, a component contains matter curves. Matter² with non-trivial chirality arises in F-theory models considered in this note. We discuss gauge theories and matter contents in F-theory compactified on such Calabi–Yau 4-folds. In particular, we investigate the gauge theories in F-theory models without a global section in detail, which is the central subject of this study. In the two constructions of genus-one fibered Calabi–Yau 4-folds without a section, we particularly focus on the families given by specific equations. The specific equations of genus-one fibered Calabi–Yau 4-folds that we choose enable a detailed investigation of the gauge theories in F-theory models.

In this note, we take a direct approach to deduce physical information directly from the defining equations of the constructed genus-one fibered Calabi–Yau 4-folds without a section. We consider two families of hypersurfaces in a product of projective spaces, which we refer to as “Fermat-type hypersurfaces” and “hypersurfaces in Hesse form”³; one family of double covers of a product of projective spaces given by equations of a specific form. Among the

¹[20, 21] discussed F-theory compactifications without a global section.

²See, for example, [35–41] for the correspondence of the singularities of Calabi–Yau manifolds and the associated matter contents. For discussion of the deformation and the resolution of singularities of manifolds, see, for example, [42]. For analysis of matter in four-dimensional (4d) F-theory with flux, see, e.g., [43, 44].

³Similar conventions of terms were made for K3 hypersurfaces in [33].

families of genus-one fibered Calabi–Yau 4-folds without a global section that we consider in this study, genus-one fibers of Fermat-type hypersurfaces and double covers of a product of projective spaces (given by equations of specific forms) possess particular symmetries; these symmetries of genus-one fibers strictly limit possible monodromies around the singular fibers. Consequently, these symmetries greatly constrain possible non-Abelian gauge groups that can form on the 7-branes. We deduce the non-Abelian gauge symmetries arising on the 7-branes in F-theory models, and utilizing these constraints imposed by the symmetries of genus-one fibers, we perform a consistency check of our results.⁴

Concretely, we consider multidegree (3,2,2,2) hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, and double covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ ramified over a multidegree (4,4,4,4) 3-fold. We find that, in F-theory compactifications on Fermat-type (3,2,2,2) hypersurfaces, generically $SU(3)$ gauge symmetries arise on the 7-branes, and when the 7-branes coincide, $SU(3)$ symmetries on the 7-branes collide and are enhanced to E_6 symmetry. Only gauge symmetries of type $SU(N)$ arise on the 7-branes in F-theory compactifications on (3,2,2,2) hypersurfaces in Hesse form. In F-theory compactifications on double covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ ramified over a multidegree (4,4,4,4) 3-fold (given by equations of specific form), generically $SU(2)$ gauge symmetries arise on the 7-branes. When the 7-branes coincide, $SU(2)$ gauge symmetries collide and are enhanced to $SO(8)$ symmetry; when more 7-branes coincide, gauge symmetries are enhanced further to E_7 symmetry.

We compute the Jacobian fibrations of the families of genus-one fibered Calabi–Yau 4-folds without a global section. We determine the Mordell–Weil groups of the Jacobian fibrations of specific members of the family of Fermat-type hypersurfaces, and the family of double covers. In particular, we deduce $U(1)$ gauge symmetries in F-theory compactifications on these members.

We also discuss potential matter contents and potential Yukawa couplings. As will be discussed in Section 4, we need to consider intrinsic 2-cycles as candidates for four-form fluxes⁵, and we need to compute their self-intersections to see if they can cancel the tadpole; however, it is technically difficult to compute the self-intersection of an intrinsic 2-cycle in the geometry of Calabi–Yau 4-folds that we consider in this note. We only deduce the potential matter contents, and potential Yukawa couplings. We compute the Euler characteristics of the constructed Calabi–Yau 4-folds, to derive constraints imposed on the self-intersection of a four-form flux to cancel the tadpole.

The outline of this note is as follows: In Section 2, we introduce the two constructions of genus-one fibered Calabi–Yau 4-folds without a section. The constructions use hypersurfaces in a product of projective spaces, and double covers of a product of projective spaces; to perform a detailed study of gauge theories, we only consider families given by specific equations in these constructions. We determine the discriminant loci and their components. We describe the forms of the discriminant components. In Section 3, we deduce the non-Abelian gauge symmetries arising on the 7-branes in F-theory compactifications on the families of genus-

⁴Similar organizations to check the consistency of solutions can be found in [33, 34].

⁵Four-form flux and a generated superpotential were studied in [45]. See, for example, [46–58, 31, 59, 60] for recent progress of four-form flux in F-theory.

one fibered Calabi–Yau 4-folds lacking a global section, as introduced in Section 2. We choose the defining equations of these families of Calabi–Yau 4-folds, so that genus-one fibers possess complex multiplications of specific orders. These particular symmetries constrain possible non-Abelian gauge groups that can form on 7-branes. We confirm that the non-Abelian gauge groups that we deduce are in agreement with these constraints. This gives a consistency check of our solutions. In Section 4, we consider the existence of a consistent four-form flux. We compute the Euler characteristics of Calabi–Yau 4-folds, to derive conditions for the self-intersections of four-form fluxes to cancel the tadpole. In Section 5, we determine the potential matter spectra, and potential Yukawa couplings. In Section 6, we state our conclusions.

2 Genus-One Fibered Calabi–Yau 4-folds without a Global Section, and Discriminant Loci

In this section, we construct genus-one fibered Calabi–Yau 4-folds that lack a global section. We consider the following two constructions:

- multidegree (3,2,2,2) hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$
- double covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ branched along a multidegree (4,4,4,4) 3-fold.

These two constructions have the trivial canonical bundles $K = 0$, and they are therefore Calabi–Yau 4-folds. Furthermore, natural projections onto $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ give genus-one fibrations, so they are genus-one fibered. Additionally, they have natural projections onto $\mathbb{P}^1 \times \mathbb{P}^1$, which give K3 fibrations.

For each of these two constructions, we only consider families given by specific equations, whose symmetries allow for a detailed investigation of gauge theories. Gauge theories in F-theory on the families of Calabi–Yau 4-folds will be discussed in Section 3. In this section, we introduce the families of genus-one fibered Calabi–Yau 4-folds given by specific equations. We show that they do not admit a global section. We determine the discriminant loci of the families of Calabi–Yau 4-folds, and we describe the forms of the discriminant components.

2.1 Multidegree (3,2,2,2) Hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

2.1.1 Two Types of Equations for (3,2,2,2) Hypersurfaces

Multidegree (3,2,2,2) hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ are Calabi–Yau 4-folds. A fiber of the natural projection onto $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is a degree 3 hypersurface in \mathbb{P}^2 , which is a genus-one curve; therefore, (3,2,2,2) hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ are genus-one fibration over the base 3-fold $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. A fiber of a natural projection onto $\mathbb{P}^1 \times \mathbb{P}^1$ is a bidegree (3,2) hypersurface in $\mathbb{P}^2 \times \mathbb{P}^1$, which is a genus-one fibered K3 surface, and therefore, projection onto $\mathbb{P}^1 \times \mathbb{P}^1$ gives a K3 fibration.

In this note, we particularly focus on two families of (3,2,2,2) hypersurfaces given by the following two types of equations:

$$(t - \alpha_1)(t - \alpha_2)fX^3 + (t - \alpha_3)(t - \alpha_4)gY^3 + (t - \alpha_5)(t - \alpha_6)hZ^3 = 0 \quad (1)$$

$$(t - \beta_1)(t - \beta_2)aX^3 + (t - \beta_3)(t - \beta_4)bY^3 + (t - \beta_5)(t - \beta_6)cZ^3 - 3(t - \beta_7)(t - \beta_8)dXYZ = 0. \quad (2)$$

$[X : Y : Z]$ is homogeneous coordinates on \mathbb{P}^2 , and t is the inhomogeneous coordinate on the first \mathbb{P}^1 in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. α_i ($i = 1, \dots, 6$) and β_j ($j = 1, \dots, 8$) are points in this first \mathbb{P}^1 . f, g, h and a, b, c, d are bidegree (2,2) polynomials on $\mathbb{P}^1 \times \mathbb{P}^1$, where the \mathbb{P}^1 's in the product $\mathbb{P}^1 \times \mathbb{P}^1$ are the last two \mathbb{P}^1 's in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

We refer to the family of hypersurfaces given by the first type equation (1) as *Fermat-type* hypersurfaces, and we refer to the family of hypersurfaces given by the second type equation (2) as hypersurfaces in *Hesse form*.

For Fermat-type hypersurface (1), a K3 fiber of the projection onto the product $\mathbb{P}^1 \times \mathbb{P}^1$ of the second and third \mathbb{P}^1 's is described by the following equation:

$$(t - \alpha_1)(t - \alpha_2)X^3 + (t - \alpha_3)(t - \alpha_4)Y^3 + (t - \alpha_5)(t - \alpha_6)Z^3 = 0 \quad (3)$$

This is Fermat-type K3 hypersurface, which is discussed in [33]. Similarly, for the hypersurface in Hesse form (2), a K3 fiber of the projection onto the product $\mathbb{P}^1 \times \mathbb{P}^1$ of the second and third \mathbb{P}^1 's is given by the following equation:

$$(t - \beta_1)(t - \beta_2)X^3 + (t - \beta_3)(t - \beta_4)Y^3 + (t - \beta_5)(t - \beta_6)Z^3 - 3(t - \beta_7)(t - \beta_8)XYZ = 0. \quad (4)$$

This is K3 hypersurface in Hesse form, which is discussed in [33].

In [33], it was shown that Fermat-type K3 hypersurfaces (1) and K3 hypersurfaces in Hesse form (2) are genus-one fibered, but their generic members lack a global section to the fibration. We therefore conclude that Fermat-type (3,2,2,2) Calabi–Yau hypersurfaces (1) and Calabi–Yau hypersurfaces in Hesse form (2) are genus-one fibered, but they lack a rational section.

2.1.2 Discriminant Locus and Forms of Discriminant Components of Fermat-type (3,2,2,2) Hypersurfaces

We determine the discriminant locus, and the forms of the components of Fermat-type (3,2,2,2) hypersurface

$$(t - \alpha_1)(t - \alpha_2)fX^3 + (t - \alpha_3)(t - \alpha_4)gY^3 + (t - \alpha_5)(t - \alpha_6)hZ^3 = 0 \quad (5)$$

We find from the equation (5) that the Fermat-type hypersurface develops singularities exactly when the coefficient of X^3 , Y^3 or Z^3 vanishes. Therefore, the loci given by the following

equations in the base 3-fold $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ describe the discriminant locus:

$$\begin{aligned} t &= \alpha_i \quad (i = 1, \dots, 6) \\ f &= 0 \\ g &= 0 \\ h &= 0. \end{aligned} \tag{6}$$

Each equation in (6) gives a discriminant component. We use the following notations to denote the discriminant components:

$$\begin{aligned} A_i &:= \{t = \alpha_i\} \quad (i = 1, \dots, 6) \\ B_1 &:= \{f = 0\} \\ B_2 &:= \{g = 0\} \\ B_3 &:= \{h = 0\}. \end{aligned} \tag{7}$$

We require that

$$B_1 \cap B_2 \cap B_3 = \emptyset \tag{8}$$

to ensure that the Calabi–Yau condition is unbroken.

Component A_i , $i = 1, \dots, 6$, is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The bidegree (2,2) curve in $\mathbb{P}^1 \times \mathbb{P}^1$ is a curve of genus 1, i.e., an elliptic curve Σ_1 , and therefore, component B_i , $i = 1, 2, 3$, is isomorphic to $\mathbb{P}^1 \times \Sigma_1$.

Next, we determine the intersections of discriminant components; in other words, we find the forms of matter curves that discriminant components contain. When $\alpha_i \neq \alpha_j$, A_i and A_j are parallel. Intersection $A_i \cap B_j$ is a genus-one curve Σ_1 . Two bidegree (2,2) curves in $\mathbb{P}^1 \times \mathbb{P}^1$ meet at 8 points, and therefore, $B_i \cap B_j$, $i \neq j$, is a sum of parallel 8 rational curves \mathbb{P}^1 . We summarize the forms of discriminant components and their intersections in Table 1 below.

Component	Topology
A_i	$\mathbb{P}^1 \times \mathbb{P}^1$
B_i	$\mathbb{P}^1 \times \Sigma_1$
Intersections	
$A_i \cap B_j$	Σ_1
$B_i \cap B_j$	parallel 8 \mathbb{P}^1 's

Table 1: Discriminant components of the discriminant locus of Fermat-type hypersurface, and their intersections.

2.1.3 Discriminant Locus and Forms of Discriminant Components of (3,2,2,2) Hypersurfaces in Hesse Form

We determine the discriminant locus and the forms of the discriminant components of (3,2,2,2) hypersurface in Hesse form

$$(t - \beta_1)(t - \beta_2)aX^3 + (t - \beta_3)(t - \beta_4)bY^3 + (t - \beta_5)(t - \beta_6)cZ^3 - 3(t - \beta_7)(t - \beta_8)dXYZ = 0. \quad (9)$$

We require that all four polynomials $\{a, b, c, d\}$ do not have simultaneous zero, to preserve the Calabi–Yau condition. We also assume that $\beta_7, \beta_8 \neq \beta_i, i = 1, \dots, 6$.

We use the following notations

$$\begin{aligned} A &:= (t - \beta_1)(t - \beta_2)a \\ B &:= (t - \beta_3)(t - \beta_4)b \\ C &:= (t - \beta_5)(t - \beta_6)c \\ D &:= (t - \beta_7)(t - \beta_8)d, \end{aligned} \quad (10)$$

and the notation

$$F_{Hesse} := (t - \beta_1)(t - \beta_2)aX^3 + (t - \beta_3)(t - \beta_4)bY^3 + (t - \beta_5)(t - \beta_6)cZ^3 - 3(t - \beta_7)(t - \beta_8)dXYZ. \quad (11)$$

Genus-one fiber degenerates exactly when the equations

$$\partial_X F_{Hesse} = \partial_Y F_{Hesse} = \partial_Z F_{Hesse} = 0 \quad (12)$$

have a solution for $[X : Y : Z] \in \mathbb{P}^2$.

From this and by comparing degrees, we obtain the discriminant of the equation (9), as follows:

$$\Delta := ABC(ABC - D^3)^3 \quad (13)$$

The discriminant (13) may be rewritten explicitly as

$$\Delta = \Pi_{i=1}^6 (t - \beta_i) \cdot abc \cdot [\Pi_{i=1}^6 (t - \beta_i) \cdot abc - (t - \beta_7)^3 (t - \beta_8)^3 d^3]^3. \quad (14)$$

We use the notation

$$e := \Pi_{i=1}^6 (t - \beta_i) \cdot abc - (t - \beta_7)^3 (t - \beta_8)^3 d^3 \quad (15)$$

for simplicity. The vanishing of the discriminant $\Delta = 0$ describes the discriminant locus. Therefore, the following equations describe the discriminant components:

$$\begin{aligned} t &= \beta_i \quad (i = 1, \dots, 6) \\ a &= 0 \\ b &= 0 \\ c &= 0 \\ e &= 0. \end{aligned} \quad (16)$$

We use the following notations to denote the discriminant components:

$$\begin{aligned}
A_i &:= \{t = \beta_i\} \quad (i = 1, \dots, 6) \\
B_1 &:= \{a = 0\} \\
B_2 &:= \{b = 0\} \\
B_3 &:= \{c = 0\} \\
B_4 &:= \{e = 0\}.
\end{aligned} \tag{17}$$

Component A_i is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The bidegree (2,2) curve in $\mathbb{P}^1 \times \mathbb{P}^1$ is a genus-one curve Σ_1 , and therefore, components B_1, B_2 and B_3 are isomorphic to $\mathbb{P}^1 \times \Sigma_1$. B_4 is some complicated complex surface. We do not discuss the form of B_4 .

When $\beta_i \neq \beta_j$, components A_i and A_j are parallel. Intersection $A_i \cap B_j$, $i = 1, \dots, 6$, $j = 1, \dots, 4$, is isomorphic to Σ_1 . $B_i \cap B_j$, $i, j = 1, 2, 3$, $i \neq j$, is a sum of 8 disjoint rational curves \mathbb{P}^1 . $B_i \cap B_4$, $i = 1, 2, 3$, is a union of 8 \mathbb{P}^1 's and 2 Σ_1 's. The forms of the discriminant components and their intersections are shown in Table 2 below.

Component	Topology
A_i	$\mathbb{P}^1 \times \mathbb{P}^1$
$B_i \ (i = 1, 2, 3)$	$\mathbb{P}^1 \times \Sigma_1$
Intersections	
$A_i \cap B_j \ (j = 1, \dots, 4)$	Σ_1
$B_i \cap B_j \ (i, j = 1, 2, 3, i \neq j)$	disjoint 8 \mathbb{P}^1 's
$B_i \cap B_4 \ (i = 1, 2, 3)$	union of 8 \mathbb{P}^1 's and 2 Σ_1 's

Table 2: Discriminant components of hypersurfaces in Hesse form, and their intersections. Form of component B_4 is omitted.

2.2 Double Covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ Ramified Along a Multidegree (4,4,4,4) 3-fold

2.2.1 Equations for Double Covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

Double covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ ramified along a multidegree (4,4,4,4) 3-fold are Calabi–Yau 4-folds. A fiber of the natural projection onto $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is a double cover of \mathbb{P}^1 branched along 4 points, which is a genus-one curve. Therefore, projection onto $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is a genus-one fibration. Additionally, a fiber of natural projection onto $\mathbb{P}^1 \times \mathbb{P}^1$ is a double cover

of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along a (4,4) curve, which is a genus-one fibered K3 surface; projection onto $\mathbb{P}^1 \times \mathbb{P}^1$ gives a K3 fibration.

In this note, we focus on the family of double covers given by the following type of equation:

$$\tau^2 = f \cdot a(t) \cdot x^4 + g \cdot b(t). \quad (18)$$

x is the inhomogeneous coordinate on the first \mathbb{P}^1 in the product $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, and t is the inhomogeneous coordinate on the second \mathbb{P}^1 . a and b are degree 4 polynomials in the variable t . f and g are bidegree (4,4) polynomials on $\mathbb{P}^1 \times \mathbb{P}^1$, where the \mathbb{P}^1 's in the product $\mathbb{P}^1 \times \mathbb{P}^1$ are the last two \mathbb{P}^1 's in the product $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. By splitting the polynomials a and b into linear factors, the equation (18) may be rewritten as:

$$\tau^2 = f \cdot \prod_{i=1}^4 (t - \alpha_i) \cdot x^4 + g \cdot \prod_{j=5}^8 (t - \alpha_j). \quad (19)$$

The fiber of the projection onto the product of the third and the fourth \mathbb{P}^1 's in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is given by the following equation:

$$\tau^2 = \prod_{i=1}^4 (t - \alpha_i) \cdot x^4 + \prod_{j=5}^8 (t - \alpha_j). \quad (20)$$

This is a genus-one fibered K3 surface discussed in [34], and it was shown in [34] that this K3 surface does not admit a global section. Therefore, we conclude that the double covers (18) do not have a rational section.

2.2.2 Discriminant Locus and Forms of Discriminant Components of Double Covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

We determine the discriminant locus and the discriminant components of double cover (18). The double cover (18) develops singularity exactly when the coefficients of equation (19) vanish. Therefore, the discriminant locus in the base $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is described by the following equations:

$$\begin{aligned} t &= \alpha_i \quad (i = 1, \dots, 8) \\ f &= 0 \\ g &= 0. \end{aligned} \quad (21)$$

Each equation in (21) gives a discriminant component. We use the following notations to denote the discriminant components:

$$\begin{aligned} A_i &:= \{t = \alpha_i\} \quad (i = 1, \dots, 8) \\ B_1 &:= \{f = 0\} \\ B_2 &:= \{g = 0\}. \end{aligned} \quad (22)$$

Discriminant component A_i , $i = 1, \dots, 8$, is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The bidegree (4,4) curve in $\mathbb{P}^1 \times \mathbb{P}^1$ is a genus 9 curve Σ_9 , and therefore, component B_i , $i = 1, 2$, is isomorphic to $\mathbb{P}^1 \times \Sigma_9$.

We determine the forms of the intersections of components. When $\alpha_i \neq \alpha_j$, A_i and A_j are parallel. $A_i \cap B_j$, $i = 1, \dots, 8$, $j = 1, 2$, is isomorphic to genus 9 curve Σ_9 . Two bidegree (4,4) curves in $\mathbb{P}^1 \times \mathbb{P}^1$ meet at 32 points, and therefore, $B_1 \cap B_2$ is the disjoint sum of 32 rational curves \mathbb{P}^1 . The forms of the discriminant components and their intersections are shown in Table 3 below.

Component	Topology
A_i	$\mathbb{P}^1 \times \mathbb{P}^1$
B_i	$\mathbb{P}^1 \times \Sigma_9$
Intersections	
$A_i \cap B_j$	Σ_9
$B_1 \cap B_2$	disjoint 32 \mathbb{P}^1 's

Table 3: Discriminant components of the double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, and their intersections.

3 Gauge Symmetries

We deduce the non-Abelian gauge symmetries that form on the 7-branes in F-theory compactifications on genus-one fibered Calabi–Yau 4-folds lacking a global section, which we constructed in Section 2. Genus-one fibers of the Fermat-type Calabi–Yau hypersurfaces (1) and double covers (18) possess complex multiplications of specific orders. These greatly limit the possible monodromies around the singular fibers, and as a result, possible types of singular fibers are also restricted. These strictly constrain the possible non-Abelian gauge groups that can form on the 7-branes. Using this fact, we check the consistency of solutions of non-Abelian gauge groups in Section 3.4.

3.1 Non-Abelian Gauge Groups and Singular Fibers

When a Calabi–Yau 4-fold has a genus-one fibration, the structures of singular fibers⁶ along the codimension one locus in the base are in essence the same as those of singular fibers of elliptic surfaces. Therefore, Kodaira’s classification [61, 62] applies to singular fibers on discriminant components. According to Kodaira’s classification, the types of singular fibers fall into two classes: i) six types II , III , IV , II^* , III^* , and IV^* ; and ii) two infinite series I_n ($n \geq 1$) and I_m^* ($m \geq 0$).

⁶See [61–69] for discussion of elliptic surfaces, elliptic fibration, and singular fibers. [70] discusses elliptic curves and the Jacobian. [71–73] discuss elliptic fibrations of 3-folds.

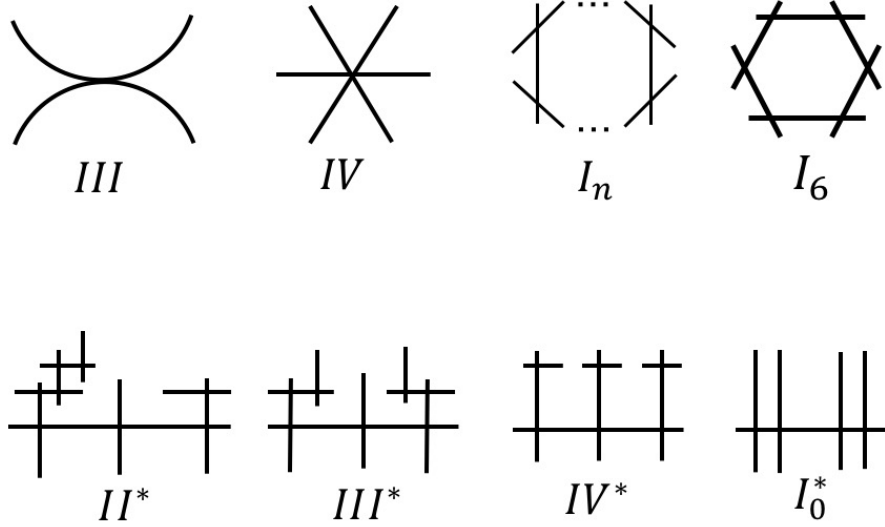


Figure 1: Singular Fibers

Fibers of type I_1 and II are rational curves \mathbb{P}^1 with one singularity (II is a rational curve with a cusp, and I_1 is a rational curve with a node); fibers of the other types are unions of smooth \mathbb{P}^1 's intersecting in specific ways. Type III fiber is a union of two rational curves tangential to each other at one point, and type IV fiber is a union of three rational curves meeting at one point. For each fiber type I_n , n rational curves intersect to form a regular n -polygon. Figure 1 shows images of the singular fibers. Each line in the image represents a rational curve \mathbb{P}^1 . Two rational curve components in a singular fiber intersect only when two lines in an image intersect.

The correspondences of the types of singular fibers on discriminant components, and non-Abelian gauge symmetries that arise on 7-branes wrapped on discriminant components, are presented in Table 4 below.

3.2 Non-Abelian Gauge Groups in F-theory on (3,2,2,2) Hyper-surfaces

We deduce non-Abelian gauge symmetries in F-theory compactification on (3,2,2,2) hyper-surfaces.

fiber type	gauge group
I_n	$SU(n)$
I_m^*	$SO(2m + 8)$
III	$SU(2)$
IV	$SU(3)$
II^*	E_8
III^*	E_7
IV^*	E_6

Table 4: Correspondence between singular fiber types and non-Abelian gauge groups.

3.2.1 Fermat-Type (3,2,2,2) Hypersurfaces

The Jacobian fibration of Fermat-type hypersurface (1) is given by the following equation:

$$X^3 + Y^3 + \Pi_{i=1}^6(t - \alpha_i) \cdot fgh \cdot Z^3 = 0. \quad (23)$$

The Jacobian fibration (23) is locally given by

$$y^2 = x^3 - 2^4 \cdot 3^3 \cdot \Pi_{i=1}^6(t - \alpha_i)^2 \cdot f^2 g^2 h^2. \quad (24)$$

Therefore, the discriminant is given by the following equation:

$$\Delta \sim \Pi_{i=1}^6(t - \alpha_i)^4 \cdot f^4 g^4 h^4. \quad (25)$$

Note that a genus-one fibered Calabi–Yau 4-fold and its Jacobian fibration have identical discriminant loci. Indeed, the discriminant locus of the Jacobian (23) is given by $\Delta = 0$, and this is identical to the discriminant locus of the Fermat-type hypersurface, which we obtained in Section 2.1.2.

By applying Tate’s algorithm [65] to equation (24), we determine the types of singular fibers. We find that, when α_i ($i = 1, \dots, 6$) are mutually distinct, the singular fiber on component A_i is of type IV . Correspondingly, $SU(3)$ gauge symmetry arises on 7-branes wrapped on component A_i . When the multiplicity of α_i is 2, (i.e. when there is one $j \neq i$ such that $\alpha_i = \alpha_j$), 7-branes wrapped on components A_i and A_j coincide, and the fiber type is enhanced to IV^* . The corresponding gauge group on 7-branes is enhanced to E_6 . To preserve Calabi–Yau condition, the multiplicity cannot be greater than 2. Type of singular fibers on component B_i is IV ; $SU(3)$ gauge symmetry arises on 7-branes wrapped on component B_i . The results are summarized in Table 5 below.

Component	Fiber type	non-Abel. Gauge Group
A_i	IV	$SU(3)$
	IV^*	E_6
B_i	IV	$SU(3)$

Table 5: Types of singular fibers and corresponding non-Abelian gauge groups on discriminant components of Fermat-type hypersurface.

3.2.2 (3,2,2,2) Hypersurfaces in Hesse Form

As we saw in Section 2.1.2, the equation for (3,2,2,2) hypersurface in Hesse form (2) has the following discriminant:

$$\Delta = \Pi_{i=1}^6(t - \beta_i) \cdot abc \cdot e^3. \quad (26)$$

Singular fibers on discriminant components are seen to be multiplicative, and therefore, each fiber type is I_n for some n . This means that non-Abelian gauge symmetries on 7-branes are $SU(N)$ for some N ; they cannot be $SO(M)$ or exceptional Lie groups E_n . We can determine the fiber type on the discriminant component from the multiplicity of zero of the discriminant (26). When β_i 's are mutually distinct, the fiber type on component A_i is I_1 , and non-Abelian gauge symmetry does not form on the 7-brane wrapped on A_i . As the multiplicity of β_i increases, more 7-branes become coincident, and the non-Abelian gauge group becomes further enhanced. The maximum enhancement occurs when all β_i , $i = 1, \dots, 6$, are equal, and all six 7-branes wrapped on A_i coincide. The fiber type on component A_1 for this case is I_6 , and $SU(6)$ gauge symmetry arises on the 7-branes wrapped on A_1 . In Section 5, we compute the potential matter spectra for this most enhanced situation.

Singular fibers on component B_i , $i = 1, 2, 3$, have type I_1 ; a non-Abelian gauge group does not form on the 7-brane wrapped on component B_i , $i = 1, 2, 3$. Singular fibers on component B_4 have type I_3 , and $SU(3)$ gauge group arises on 7-branes wrapped on B_4 . Results are summarized in Table 6.

3.3 Non-Abelian Gauge Groups in F-theory on Double Covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

We deduce the non-Abelian gauge groups in F-theory compactifications on double covers (18).

The Jacobian fibration of double cover (18) is given by the following equation [74]:

$$\tau^2 = \frac{1}{4}x^3 - fg \cdot \Pi_{i=1}^8(t - \alpha_i) \cdot x. \quad (27)$$

The discriminant of the Jacobian fibration (27) is given by

$$\Delta \sim f^3 g^3 \cdot \Pi_{i=1}^8(t - \alpha_i)^3. \quad (28)$$

Component	Fiber type	non-Abel. Gauge Group
A_i	I_1	None.
	I_2	$SU(2)$
	I_3	$SU(3)$
	I_4	$SU(4)$
	I_5	$SU(5)$
	I_6	$SU(6)$
$B_{1,2,3}$	I_1	None.
B_4	I_3	$SU(3)$

Table 6: Types of singular fibers and corresponding (non-Abelian) gauge groups on discriminant components of hypersurface in Hesse form.

The condition $\Delta = 0$ describes the discriminant locus of the Jacobian (27). We find that this is identical to the discriminant locus obtained in Section 2.2.2.

We compute the types of singular fibers on the discriminant components by applying Tate's algorithm to the Jacobian fibration (27). When α_i 's are mutually distinct, the singular fiber on component A_i has type III ; the $SU(2)$ gauge group arises on the 7-branes wrapped on component A_i for this case. When the multiplicity of α_i is 2, say there is $j \neq i$ such that $\alpha_i = \alpha_j$, then the 7-branes wrapped on components A_i and A_j become coincident, and singular fiber on component A_i has type I_0^* ; the non-Abelian gauge symmetry on the 7-branes wrapped on component A_i becomes enhanced to $SO(8)$. When the multiplicity of α_i is 3, the singular fiber on component A_i has type III^* , and the gauge symmetry on component A_i is further enhanced to E_7 . To preserve the Calabi–Yau condition, no further enhancement is possible. The singular fibers on component B_i is of type III ; the $SU(2)$ gauge group arises on 7-branes wrapped on component B_i . The results are displayed in Table 7 below.

3.4 Consistency Check by Monodromy

We consider monodromies around singular fibers to perform a consistency check of solutions of non-Abelian gauge groups, which we obtained in Sections 3.2, 3.3. Genus-one fibers of Fermat-type (3,2,2,2) hypersurfaces (1) and double covers (18) possess particular symmetries, and as a result, these symmetries strictly constrain monodromies around singular fibers. We confirm that the non-Abelian gauge symmetries obtained by us in agreement with these restrictions.

Component	Fiber type	non-Abel. Gauge Group
A_i	III	$SU(2)$
	I_0^*	$SO(8)$
	III^*	E_7
B_i	III	$SU(2)$

Table 7: Types of singular fibers and corresponding non-Abelian gauge groups on discriminant components of double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

3.4.1 Monodromy and J-invariant

Genus-one fibers of Fermat-type (3,2,2,2) hypersurfaces and double covers (18) have constant j-invariants; they are constant over the base 3-fold $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Concretely, generic genus-one fiber of the Fermat-type (3,2,2,2) hypersurface is the Fermat curve⁷, whose j-invariant is known to be 0. Therefore, the j-invariant of singular fibers is forced to be 0.

Smooth genus-one fiber of double cover (18) is invariant under the map:

$$x \rightarrow e^{2\pi i/4}x, \quad (29)$$

whose order is 4. This is a complex multiplication of order 4, and therefore, the generic genus-one fiber has the j-invariant 1728. This forces the j-invariant of singular fibers to be 1728.

Each fiber type has a specific monodromy and j-invariant. We display the monodromy and their orders in $SL_2(\mathbb{Z})$, and the j-invariant, for each fiber type in Table 8 below. “Finite” in the table means that the j-invariant of fiber type I_0^* can take any finite value in \mathbb{C} . Results in Table 8 were derived in [61, 62]⁸.

3.4.2 Fermat-Type (3,2,2,2) Hypersurfaces

Generic fibers of the Fermat-type (3,2,2,2) hypersurface are Fermat curves, and their j-invariant is 0. Therefore, singular fibers have j-invariant 0. As can be seen in Table 8, the fiber types with j-invariant 0 are only II , IV , I_0^* , IV^* , and II^* . Fiber types on discriminant components that we obtained in 3.2.1 are IV , IV^* , which is in agreement with constraint imposed by the j-invariant. Monodromies of order 3 characterize non-Abelian gauge symmetries arising on 7-branes in F-theory compactifications on Fermat-type (3,2,2,2) hypersurfaces.

⁷The Fermat curve possesses complex multiplication of order 3.

⁸Euler numbers of fiber types were obtained in [62], and they have an interpretation as the number of 7-branes wrapped on.

Fiber Type	j-invariant	Monodromy	order of Monodromy	# of 7-branes (Euler number)
I_0^*	finite	$-\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	2	6
I_b	∞	$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$	infinite	b
I_b^*	∞	$-\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$	infinite	$b+6$
II	0	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	6	2
II^*	0	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	6	10
III	1728	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	4	3
III^*	1728	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	4	9
IV	0	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	3	4
IV^*	0	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	3	8

Table 8: Fiber types, their j-invariants, monodromies, and the associated numbers of 7-branes.

3.4.3 Double Covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

The smooth genus-one fiber of double cover (18) possesses complex multiplication of order 4, generated by the map

$$x \rightarrow e^{2\pi i/4} x. \quad (30)$$

Therefore, generic fibers have constant j-invariant 1728 over the base 3-fold $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, and this forces the j-invariant of singular fibers to be 1728. According to Table 8, fiber types with j-invariant 1728 are only III , I_0^* , and III^* . This agrees with the fiber types that we obtained in 3.3 on discriminant components of double covers. Monodromies of order 2 and 4 characterize non-Abelian gauge symmetries on 7-branes in F-theory compactification on double covers (18).

3.5 Mordell–Weil Groups of Some Jacobian Fibrations $-\mathbb{Z}_2$ and \mathbb{Z}_3

We specify the Mordell–Weil groups of some Jacobian fibrations of genus-one fibered Calabi–Yau 4-folds.

We particularly consider the following special Fermat-type (3,2,2,2) hypersurface:

$$(t - \alpha_1)^2 f X^3 + (t - \alpha_2)^2 g Y^3 + (t - \alpha_3)^2 h Z^3 = 0. \quad (31)$$

The Jacobian fibration of this special Fermat-type hypersurface (31) is given by the following equation:

$$X^3 + Y^3 + (t - \alpha_1)^2(t - \alpha_2)^2(t - \alpha_3)^2 \cdot fgh \cdot Z^3 = 0. \quad (32)$$

The projection onto the last two \mathbb{P}^1 's in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ gives a K3 fibration, and picking a point in the base surface $\mathbb{P}^1 \times \mathbb{P}^1$ gives a specialization to this K3 fiber. The K3 fiber of the Jacobian fibration (32) is given by the following equation:

$$X^3 + Y^3 + (t - \alpha_1)^2(t - \alpha_2)^2(t - \alpha_3)^2 Z^3 = 0. \quad (33)$$

This is the Jacobian fibration of the Fermat-type K3 hypersurface, which is discussed in [33], with reducible fiber type E_6^3 . According to Table 2 in [75], extremal K3 surface with reducible fiber type E_6^3 is uniquely determined, and its transcendental lattice has the intersection matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. The Mordell–Weil group of this extremal K3 surface is determined in [76, 75] to be \mathbb{Z}_3 .

By considering the specialization of the Jacobian fibration (32) to its K3 fiber (33), we find that the Mordell–Weil group of the Jacobian (32) is isomorphic to that of its K3 fiber (33), which is \mathbb{Z}_3 . This shows that the Mordell–Weil group of the Jacobian fibration (32) is isomorphic to \mathbb{Z}_3 . The three constant sections $[X : Y : Z] = [1 : -\omega : 0], [1 : -\bar{\omega} : 0]$ and $[1 : -1 : 0]$ form the Mordell–Weil group \mathbb{Z}_3 . In particular, F-theory compactified on the Fermat-type (3,2,2,2) hypersurface (31) does not have a $U(1)$ gauge symmetry.

Next, we consider the double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ given by the following equation:

$$\tau^2 = f \cdot (t - \alpha_1)^3(t - \alpha_2) \cdot x^4 + g \cdot (t - \alpha_2)(t - \alpha_3)^3. \quad (34)$$

The Jacobian fibration of this double cover is given by:

$$\tau^2 = \frac{1}{4}x^3 - fg \cdot (t - \alpha_1)^3(t - \alpha_2)^2(t - \alpha_3)^3 \cdot x. \quad (35)$$

The K3 fiber of the Jacobian fibration (35) is given by the equation:

$$\tau^2 = \frac{1}{4}x^3 - (t - \alpha_1)^3(t - \alpha_2)^2(t - \alpha_3)^3 \cdot x. \quad (36)$$

This extremal K3 surface (36) has the reducible fiber type $E_7^2 D_4$. As discussed in [34], the Mordell–Weil group of this extremal K3 surface (36) is \mathbb{Z}_2 [76, 75].

As per reasoning similar to the above argument, we consider the specialization of the Jacobian fibration of the double cover (35) to its K3 fiber (36) and find that the Mordell–Weil group of the Jacobian (35) is isomorphic to that of its K3 fiber (36). Therefore, we conclude that the Mordell–Weil group of the Jacobian fibration (35) is isomorphic to \mathbb{Z}_2 . Constant sections $\{x = 0, \tau = 0\}$ and $\{x = \infty, \tau = \infty\}$ form the Mordell–Weil group \mathbb{Z}_2 . It follows that F-theory compactification on the double cover (34) does not have a $U(1)$ gauge symmetry.

4 Discussion of Consistent Four-Form Flux and Euler Characteristics

4.1 Review of Conditions on Four-Form Flux

We briefly review physical conditions imposed on four-form flux G_4 . The quantization condition [77] imposed on four-form flux is given by the following equation:

$$G_4 + \frac{1}{2}c_2(Y) \in H^4(Y, \mathbb{Z}). \quad (37)$$

In particular, when the second Chern class $c_2(Y)$ is even, the term $\frac{1}{2}c_2(Y)$ is irrelevant. To preserve supersymmetry in 4d theory, the following conditions need to be imposed [78] on four-form flux:

$$G_4 \in H^{2,2}(Y) \quad (38)$$

$$G_4 \wedge J = 0. \quad (39)$$

J in the condition (39) represents a Kähler form.

Furthermore, to ensure that the 4d effective theory has Lorentz symmetry, four-form flux is required to have one leg in the fiber [79]. When genus-one fibration admits a global section, this condition is given by the following equations:

$$G_4 \cdot \tilde{p}^{-1}(C) \cdot \tilde{p}^{-1}(C') = 0 \quad (40)$$

$$G_4 \cdot S_0 \cdot \tilde{p}^{-1}(C) = 0 \quad (41)$$

for any $C, C' \in H^{1,1}(B_3)$. B_3 denotes base 3-fold. In the equation (41), S_0 denotes a rational zero section.

Generalization of this condition to genus-one fibration without a section was proposed in [31]; the generalized equations are as follows:

$$G_4 \cdot p^{-1}(C) \cdot p^{-1}(C') = 0 \quad (42)$$

$$G_4 \cdot \hat{N} \cdot p^{-1}(C) = 0 \quad (43)$$

for any $C, C' \in H^{1,1}(B_3)$. \hat{N} is some appropriate sum of an n -section N and exceptional divisors.

The condition to cancel the tadpole, including 3-branes, is given as follows [80, 81]:

$$\frac{\chi(Y)}{24} = \frac{1}{2}G_4 \cdot G_4 + N_3. \quad (44)$$

N_3 denotes the number of 3-branes minus anti 3-branes, and the stability of compactification requires $N_3 \geq 0$.

4.2 Intrinsic Algebraic 2-cycles as Candidates for Four-Form Fluxes

We use algebraic 2-cycles as candidates for four-form fluxes. With this choice, the condition (38) is satisfied. After some computations, we find that algebraic 2-cycles that satisfy both the conditions (39) and (42) are only *intrinsic* algebraic 2-cycles. The conditions (39) and (42) rule out all 2-cycles obtained as the restrictions of 2-cycles in the ambient space $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ to the (3,2,2,2) hypersurface, and the pullbacks of 2-cycles in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ to the double cover.

In Calabi–Yau 4-folds that we constructed, however, it is considerably difficult to explicitly describe intrinsic 2-cycles. Consequently, it is difficult to compute the self-intersections of intrinsic 2-cycles in constructed Calabi–Yau 4-folds. We do not discuss whether a consistent four-form flux exists. In Section 4.3 below, we compute the Euler characteristics of Calabi–Yau 4-folds, to derive conditions on the self-intersection of four-form flux to cancel the tadpole.

4.3 Euler Characteristics and Self-Intersection of Four-Form Flux to Cancel Tadpole

4.3.1 Multidegree (3,2,2,2) Hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

We compute the Euler characteristic of a multidegree (3,2,2,2) hypersurface Y in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. We have the following exact sequence of bundles:

$$0 \longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{T}_{\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}|_Y \longrightarrow \mathcal{N}_Y \longrightarrow 0. \quad (45)$$

\mathcal{T}_Y is the tangent bundle of a genus-one fibered Calabi–Yau multidegree (3,2,2,2) hypersurface Y , and this naturally embeds into the tangent bundle $\mathcal{T}_{\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}$ of the ambient space $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. $|_Y$ means the restriction to Y . \mathcal{N}_Y is the resultant normal bundle. We have

$$\mathcal{N}_Y \cong \mathcal{O}(3, 2, 2, 2). \quad (46)$$

From the exact sequence (45), we obtain

$$c(\mathcal{T}_Y) = \frac{c(\mathcal{T}_{\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1})|_Y}{c(\mathcal{N}_Y)}. \quad (47)$$

We have

$$c(\mathcal{T}_{\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1})|_Y = (1 + 3x + 3x^2)(1 + 2y)(1 + 2z)(1 + 2w)|_Y, \quad (48)$$

and

$$c(\mathcal{N}_Y) = 1 + 3x + 2y + 2z + 2w. \quad (49)$$

From equations (47), (48), and (49), we can compute $c(\mathcal{T}_Y)$. The top Chern class of $c(\mathcal{T}_Y)$ gives the Euler characteristic of (3,2,2,2) Calabi–Yau hypersurface Y . Therefore, we find that

$$\chi(Y) = 1584, \quad (50)$$

and

$$\frac{\chi(Y)}{24} = 66. \quad (51)$$

We also obtain the second Chern class $c_2(Y)$ from (47):

$$c_2(Y) = (3x^2 + 6xy + 6xz + 6xw + 4yz + 4zw + 4wy)|_Y. \quad (52)$$

From this, we see that the second Chern class $c_2(Y)$ is not even.

From (50), we obtain the net number of 3-branes N_3 needed to cancel the tadpole as:

$$\begin{aligned} N_3 &= \frac{\chi(Y)}{24} - \frac{1}{2}G_4 \cdot G_4 \\ &= 66 - \frac{1}{2}G_4 \cdot G_4. \end{aligned} \quad (53)$$

This must be a non-negative integer, and we therefore obtain a numerical bound on the self-intersection of a four-form flux G_4 :

$$66 \geq \frac{1}{2}G_4 \cdot G_4. \quad (54)$$

Notice that the result (50) of the Euler characteristic is valid for any (3,2,2,2) hypersurface in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. In particular, (50) gives the Euler characteristics of both the Fermat-type hypersurface and the hypersurface in Hesse form.

4.3.2 Double Covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ Ramified Along a Multidegree (4,4,4,4) 3-fold

We compute the Euler characteristic of double cover Y of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ branched along a (4,4,4,4) 3-fold B . The Euler characteristic $\chi(Y)$ of a double cover Y is given by

$$\chi(Y) = 2 \cdot \chi(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) - \chi(B). \quad (55)$$

We have

$$\chi(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) = 2^4 = 16, \quad (56)$$

therefore

$$\chi(Y) = 32 - \chi(B). \quad (57)$$

We use the exact sequence:

$$0 \longrightarrow \mathcal{T}_B \longrightarrow \mathcal{T}_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}|_B \longrightarrow \mathcal{N}_B \longrightarrow 0 \quad (58)$$

to obtain the equality

$$c(\mathcal{T}_B) = \frac{c(\mathcal{T}_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1})|_B}{c(\mathcal{N}_B)}. \quad (59)$$

$$\mathcal{N}_B \cong \mathcal{O}(4, 4, 4, 4), \quad (60)$$

therefore

$$c(\mathcal{N}_B) = 1 + 4x + 4y + 4z + 4w. \quad (61)$$

We have

$$c(\mathcal{T}_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1})|_B = (1 + 2x)(1 + 2y)(1 + 2z)(1 + 2w)|_B. \quad (62)$$

From the equality (59), we can compute $c(B)$. The top Chern class of $c(B)$ gives the Euler characteristic $\chi(B)$. Therefore, we deduce that

$$\chi(B) = -3712. \quad (63)$$

We finally obtain the Euler characteristic $\chi(Y)$:

$$\chi(Y) = 32 - \chi(B) = 32 - (-3712) = 3744. \quad (64)$$

This is divisible by 24:

$$\frac{\chi(Y)}{24} = 156. \quad (65)$$

The net number of 3-branes N_3 needed to cancel the tadpole is

$$\begin{aligned} N_3 &= \frac{\chi(Y)}{24} - \frac{1}{2}G_4 \cdot G_4 \\ &= 156 - \frac{1}{2}G_4 \cdot G_4. \end{aligned} \quad (66)$$

N_3 must be a non-negative integer, and therefore, a bound on the self-intersection of four-form flux G_4 that we obtain is

$$156 \geq \frac{1}{2}G_4 \cdot G_4. \quad (67)$$

5 Matter Spectra and Yukawa Couplings

We discuss matter fields arising on components and along matter curves. As discussed in [44], suppose gauge group G on 7-branes breaks to a subgroup Γ such that

$$\Gamma \times H \subset G \quad (68)$$

is maximal. This corresponds to the deformation of singularity associated with gauge group G , and consequently, matter fields arise on 7-branes [38]. When $\Gamma \times H$ has a representation (τ, T) , matter fields arise in representation τ of Γ , and its generation is given by [44]

$$n_\tau - n_{\tau^*} = - \int_S c_1(S)c_1(\mathcal{T}). \quad (69)$$

S denotes a component of the discriminant locus on which 7-branes are wrapped, and \mathcal{T} denotes a bundle transforming in representation T of H . We consider the case in which H is $U(1)$. Let \mathcal{L} be a supersymmetric line bundle on component S .

We discuss matter contents in F-theory compactifications on families of (3,2,2,2) hypersurfaces and double covers below. We focus on specific discriminant components whose forms are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Supersymmetric line bundles on these components are isomorphic to $\mathcal{O}(a, b)$ for some integers a and b , $a, b \in \mathbb{Z}$; for line bundles to be supersymmetric, the integers a and b are subject to the condition $ab < 0$ [44].

As discussed in [44], Yukawa couplings arise from the following three cases:

- interaction of three matter fields on a bulk component
- interaction of a field on a bulk component and two matter fields localized along a matter curve, and
- triple intersection of three matter curves meeting in a point

Components we consider below have forms isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, which is a Hirzebruch surface. Therefore, Yukawa coupling does not arise from the first case [44]. Triple intersection of three matter curves does not occur in most of components that we consider below. We mainly consider Yukawa couplings arising from the second case.

As stated in Section 4, the existence of a consistent four-form flux is undetermined for Calabi–Yau genus-one fibrations constructed in this note. We can only say that matter contents and Yukawa couplings that we obtain below *could* arise.

5.1 Matter Spectra for Fermat-Type (3,2,2,2) Hypersurfaces

We compute matter spectra in F-theory compactifications on Fermat-type (3,2,2,2) hypersurfaces. We focus on discriminant component A_1 . The form of component A_1 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

We first consider the case in which components A_1 and A_2 are coincident. We abbreviate A_1 to A . Singular fibers on A have type IV^* , and E_6 gauge group arises on the 7-branes wrapped on A . We consider the case in which E_6 breaks to $SO(10)$:

$$E_6 \supset SO(10) \times U(1). \quad (70)$$

78 of E_6 decomposes under (70) as [82]:

$$\mathbf{78} = \mathbf{45}_0 + \mathbf{16}_{-3} + \overline{\mathbf{16}}_3 + \mathbf{1}_0. \quad (71)$$

Thus, matter fields **16** (could) arise on the bulk A . The generation number of matter fields **16** on the bulk A is

$$\begin{aligned} n_{\mathbf{16}} - n_{\overline{\mathbf{16}}} &= - \int_A c_1(A) c_1(\mathcal{L}^{-3}) = 3(2x + 2y)(ax + by) \\ &= 6(a + b). \end{aligned} \quad (72)$$

$A \cap B_i = \Sigma_1$, and therefore, the bulk A contains a matter curve Σ_1 , which is a genus-one curve. Since there are three components B_1 , B_2 and B_3 , the bulk A contains three matter curves Σ_1 . Along matter curve Σ_1 , **27** of E_6 decomposes under (70) as:

$$\mathbf{27} = \mathbf{16}_1 + \mathbf{10}_{-2} + \mathbf{1}_4. \quad (73)$$

Therefore, either matter fields **16** or **10** localize along a matter curve Σ_1 .

We compute the number of matter fields localized along matter curve Σ_1 . Since matter curve $A \cap B_i = \Sigma_1$ is a bidegree (2,2) curve in $\mathbb{P}^1 \times \mathbb{P}^1$, the restriction \mathcal{L}_{Σ_1} of the line bundle $\mathcal{L} \cong \mathcal{O}(a, b)$ to matter curve $A \cap B_i = \Sigma_1$ is

$$\mathcal{L}_{\Sigma_1} \cong \mathcal{O}_{\Sigma_1}(V) \quad (74)$$

for some divisor V with $\deg V = 2(a + b)$. We have

$$n_{\mathbf{16}} = h^0(K_{\Sigma_1}^{1/2} \otimes \mathcal{O}_{\Sigma_1}(V)) = h^0(\mathcal{O}_{\Sigma_1}(V)), \quad (75)$$

and by the Riemann–Roch theorem,

$$n_{\mathbf{16}} = h^0(\mathcal{O}_{\Sigma_1}(V)) = \begin{cases} \deg V = 2(a + b) & (a + b \geq 0) \\ 0 & (a + b < 0) \end{cases} \quad (76)$$

Similarly,

$$\begin{aligned} n_{\mathbf{10}} &= h^0(K_{\Sigma_1}^{1/2} \otimes \mathcal{L}_{\Sigma_1}^{-2}) \\ &= h^0(\mathcal{O}_{\Sigma_1}(-2V)) \\ &= \begin{cases} \deg(-2V) = -4(a + b) & (a + b < 0) \\ 0 & (a + b \geq 0) \end{cases} \end{aligned} \quad (77)$$

From the above results, we see that only either **16** or **10** (could) arise along matter curve Σ_1 , depending on the sign of $a + b$. When $a + b > 0$, matter fields **16** arise on A , and matters **16** are localized along matter curve Σ_1 . Yukawa coupling for this case is

$$\mathbf{16} \cdot \mathbf{16} \cdot \mathbf{16}. \quad (78)$$

When $a + b < 0$, matter fields on A are $\overline{\mathbf{16}}$, and matters localized along matter curve Σ_1 are **10**. Yukawa coupling for this case is

$$\overline{\mathbf{16}} \cdot \mathbf{10} \cdot \mathbf{10}. \quad (79)$$

Next, we consider the case where component A_1 is not coincident with any other component A_i for $i \neq 1$. Then, the singular fibers on component A_1 have type *IV*. We again abbreviate A_1 to A . $SU(3)$ gauge group arises on the 7-branes wrapped on the bulk A . When $SU(3)$ breaks to $SU(2)$ with

$$SU(3) \supset SU(2) \times U(1), \quad (80)$$

8 of $SU(3)$ decomposes as:

$$\mathbf{8} = \mathbf{3}_0 + \mathbf{2}_3 + \overline{\mathbf{2}}_{-3} + \mathbf{1}_0. \quad (81)$$

Therefore, matter fields **2** (could) arise on the bulk A . **3** of $SU(3)$ decomposes as:

$$\mathbf{3} = \mathbf{2}_1 + \mathbf{1}_{-2}. \quad (82)$$

Therefore, we see that matter fields **2** (or singlets) could localize along the matter curve $A \cap B_i = \Sigma_1$.

The number of generation of matter fields **2** on component A is given by:

$$\begin{aligned} n_{\mathbf{2}} - n_{\bar{\mathbf{2}}} &= - \int_A c_1(A) c_1(\mathcal{L}^3) = -3(2x + 2y)(ax + by) \\ &= -6(a + b). \end{aligned} \quad (83)$$

As we saw the above, the restriction \mathcal{L}_{Σ_1} of the line bundle \mathcal{L} to the matter curve Σ_1 is

$$\mathcal{L}_{\Sigma_1} \cong \mathcal{O}_{\Sigma_1}(V) \quad (84)$$

for some divisor V of degree $2(a + b)$. Now by the Riemann-Roch theorem,

$$\begin{aligned} n_{\mathbf{2}} &= h^0(K_{\Sigma_1}^{1/2} \otimes \mathcal{L}_{\Sigma_1}) \\ &= h^0(\mathcal{O}_{\Sigma_1}(V)) \\ &= \begin{cases} \deg V = 2(a + b) & (a + b \geq 0) \\ 0 & (a + b < 0) \end{cases} \end{aligned} \quad (85)$$

Similarly,

$$\begin{aligned} n_{\mathbf{1}} &= h^0(K_{\Sigma_1}^{1/2} \otimes \mathcal{L}_{\Sigma_1}^{-2}) \\ &= h^0(\mathcal{O}_{\Sigma_1}(-2V)) \\ &= \begin{cases} \deg(-2V) = -4(a + b) & (a + b \leq 0) \\ 0 & (a + b > 0) \end{cases} \end{aligned} \quad (86)$$

Therefore, when $a + b > 0$, the number of generations $n_{\mathbf{2}} - n_{\bar{\mathbf{2}}}$ on the bulk A is negative, and matter fields $\bar{\mathbf{2}}$ arise on the bulk A , and matter fields **2** are localized along matter curve Σ_1 . Yukawa coupling for this case is

$$\bar{\mathbf{2}} \cdot \mathbf{2} \cdot \mathbf{2}. \quad (87)$$

When $a + b < 0$, matter fields **2** arise on A , and singlets **1** are localized along matter curve Σ_1 . Yukawa coupling is given by

$$\mathbf{2} \cdot \mathbf{1} \cdot \mathbf{1}. \quad (88)$$

Notice that we have three matter curves Σ_1 in the bulk A , and these are the intersections $A \cap B_1$, $A \cap B_2$ and $A \cap B_3$. As stated in Section 2.1.2, we require the following condition:

$$B_1 \cap B_2 \cap B_3 = \phi, \quad (89)$$

to preserve Calabi–Yau condition. Owing to this requirement, the 3 matter curves do not have a triple intersection. Therefore, Yukawa coupling does not arise from the interaction of three matter fields along these three matter curves.

The results are shown in Table 9 below.

Gauge Group	$a + b$	Matter on E	# Gen. on E	Matter on Σ_1	# Gen. on Σ_1	Yukawa
E_6	> 0	16	$6(a + b)$	16	$2(a + b)$	16 · 16 · 16
	< 0	$\overline{\mathbf{16}}$	$-6(a + b)$	10	$-4(a + b)$	$\overline{\mathbf{16}} \cdot \mathbf{10} \cdot \mathbf{10}$
$SU(3)$	> 0	$\overline{\mathbf{2}}$	$6(a + b)$	2	$2(a + b)$	$\overline{\mathbf{2}} \cdot \mathbf{2} \cdot \mathbf{2}$
	< 0	2	$-6(a + b)$	1	$-4(a + b)$	$\mathbf{2} \cdot \mathbf{1} \cdot \mathbf{1}$

Table 9: Potential matter spectra for Fermat-type hypersurface.

5.2 Matter Spectra for (3,2,2,2) Hypersurfaces in Hesse Form

We compute matter spectra in F-theory compactifications on (3,2,2,2) hypersurfaces in Hesse Form. We focus on components A_1 , and we consider the extreme case in which all six components $\{A_i\}_{i=1}^6$ are coincident. A_1 is abbreviated to A below. In this case, singular fibers on the bulk A have type I_6 , and the $SU(6)$ gauge group arises on the 7-branes wrapped on A . The form of A is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

When $SU(6)$ breaks to $SU(5)$ with

$$SU(6) \supset SU(5) \times U(1), \quad (90)$$

the adjoint **35** of $SU(6)$ decomposes as

$$\mathbf{35} = \mathbf{24}_0 + \mathbf{5}_6 + \overline{\mathbf{5}}_{-6} + \mathbf{1}_0. \quad (91)$$

Therefore, matter fields **5** (could) arise on the bulk A . The generation of matter fields **5** on the bulk A is given by:

$$n_{\mathbf{5}} - n_{\overline{\mathbf{5}}} = - \int_A c_1(A) c_1(\mathcal{L}^6) = -12(a + b). \quad (92)$$

$A \cap B_i = \Sigma_1$, $i = 1, 2, 3, 4$, and therefore, the bulk A contains four matter curves Σ_1 . Under

$$SU(7) \supset SU(6) \times U(1), \quad (93)$$

adjoint **48** of $SU(7)$ decomposes as

$$\mathbf{48} = \mathbf{35}_0 + \mathbf{6}_{-7} + \overline{\mathbf{6}}_7 + \mathbf{1}_0. \quad (94)$$

From this, we find that matter fields **6** localize along matter curve Σ_1 , before the supersymmetric line bundle \mathcal{L} is turned on over the bulk A . When the supersymmetric line bundle \mathcal{L} is turned on, **6** of $SU(6)$ along matter curve Σ_1 decomposes as

$$\mathbf{6} = \mathbf{5}_1 + \mathbf{1}_{-5}. \quad (95)$$

Therefore, the mater fields $\mathbf{5}$ (or singlets) could localize along a matter curve Σ_1 .

As per reasoning similar to computation in Section 5.1, we find that the restriction \mathcal{L}_{Σ_1} of the line bundle \mathcal{L} to matter curve Σ_1 is $\mathcal{L}_{\Sigma_1} \cong \mathcal{O}_{\Sigma_1}(V)$ for some divisor V of degree $2(a+b)$. We have

$$\begin{aligned} n_{\mathbf{5}} &= h^0(K_{\Sigma_1}^{1/2} \otimes \mathcal{L}_{\Sigma_1}) \\ &= h^0(\mathcal{O}_{\Sigma_1}(V)). \end{aligned} \quad (96)$$

By the Riemann–Roch theorem,

$$n_{\mathbf{5}} = \begin{cases} \deg V = 2(a+b) & (a+b > 0) \\ 0 & (a+b \leq 0) \end{cases} \quad (97)$$

Similarly,

$$\begin{aligned} n_{\mathbf{1}} &= h^0(-5V) \\ &= \begin{cases} \deg(-5V) = -10(a+b) & (a+b < 0) \\ 0 & (a+b \geq 0) \end{cases} \end{aligned} \quad (98)$$

Therefore, when $a+b > 0$ mater fields $\bar{\mathbf{5}}$ arise on the bulk A , and matter fields $\mathbf{5}$ localize along matter curve Σ_1 . For this case, Yukawa coupling that arises is

$$\bar{\mathbf{5}} \cdot \mathbf{5} \cdot \mathbf{5}. \quad (99)$$

When $a+b < 0$, matter fields $\mathbf{5}$ arise on the bulk A , and singlets $\mathbf{1}$ localize along matter curve Σ_1 . Yukawa coupling for this case is

$$\mathbf{5} \cdot \mathbf{1} \cdot \mathbf{1}. \quad (100)$$

The results are shown in Table 10 below.

Gauge Group	$a+b$	Matter on E	# Gen. on E	Matter on Σ_1	# Gen. on Σ_1	Yukawa
$SU(6)$	> 0	$\bar{\mathbf{5}}$	$12(a+b)$	$\mathbf{5}$	$2(a+b)$	$\bar{\mathbf{5}} \cdot \mathbf{5} \cdot \mathbf{5}$
	< 0	$\mathbf{5}$	$-12(a+b)$	$\mathbf{1}$	$-10(a+b)$	$\mathbf{5} \cdot \mathbf{1} \cdot \mathbf{1}$

Table 10: Potential matter spectra for hypersurface in Hesse form.

Triple intersection of three matter curves in the bulk E occurs for hypersurface in Hesse form. For example, we may consider the triple intersection of $\{a=0\}$, $\{b=0\}$ and $\{c=0\}$. For generic polynomials a, b, c , triple intersection does not occur, and we therefore have to make some special choice. $\{a=0\}$, $\{b=0\}$ meet in 8 points in $\mathbb{P}^1 \times \mathbb{P}^1$. When we choose

specific polynomial c so that $\{c = 0\}$ passes through one of these 8 points, the three matter curves Σ_1 , $A \cap B_1$, $A \cap B_2$ and $A \cap B_3$, meet in one point in the bulk A .

Since every fiber is multiplicative in Hesse form hypersurfaces, we expect that on the triple intersection the singularity type is A_8 , corresponding to $I_{6+1+1+1}$. Then adjoint **80** of $SU(9)$ decomposes under

$$SU(9) \supset SU(8) \times U(1) \supset SU(6) \times SU(2) \times U(1)^2 \quad (101)$$

as

$$\mathbf{80} = (\mathbf{35}, \mathbf{1}) + (\mathbf{6}, \bar{\mathbf{2}}) + (\bar{\mathbf{6}}, \mathbf{2}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{1}, \mathbf{1}) + (\mathbf{6}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}) + (\bar{\mathbf{6}}, \mathbf{1}) + (\mathbf{1}, \bar{\mathbf{2}}) + (\mathbf{1}, \mathbf{1}). \quad (102)$$

From this, we find that matter fields **6** localize along two out of three matter curves Σ_1 , and matter fields $\bar{\mathbf{6}}$ localize along the remaining one matter curve. When the supersymmetric line bundle \mathcal{L} is turned on, **6** decomposes as

$$\mathbf{6} = \mathbf{5}_1 + \mathbf{1}_{-5}. \quad (103)$$

Similarly, $\bar{\mathbf{6}}$ decomposes as

$$\bar{\mathbf{6}} = \bar{\mathbf{5}}_{-1} + \mathbf{1}_5. \quad (104)$$

Therefore, when $a + b > 0$, the Yukawa coupling

$$\mathbf{5} \cdot \mathbf{5} \cdot \mathbf{1} \quad (105)$$

arises from the triple intersection of three matter curves Σ_1 .

For the case $a + b < 0$, the Yukawa coupling

$$\bar{\mathbf{5}} \cdot \mathbf{1} \cdot \mathbf{1} \quad (106)$$

arises from the triple intersection of three matter curves.

5.3 Matter Spectra for Double Covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ Branched Along a Multidegree (4,4,4,4) 3-fold

We compute matter spectra in F-theory compactifications on double covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ branched along a multidegree (4,4,4,4) 3-fold (18).

Singular fibers on components B_1 , B_2 have type III , and $SU(2)$ gauge groups arise on the 7-branes wrapped on these components. Therefore, we see that matter does not arise on the 7-branes wrapped on B_1 and B_2 . Similarly, when A_1 is not coincident with any other A_i , $i \neq 1$, singular fibers on A_1 have type III . For this situation, matter does not arise on the 7-branes wrapped on A_1 .

When A_1 is coincident with another A_i , say $A_1 = A_2$, $SO(8)$ gauge group arises on the 7-branes wrapped on A_1 . A_1 is abbreviated to A . When gauge group $SO(8)$ breaks to $SU(4)$ under

$$SO(8) \supset SU(4) \times U(1), \quad (107)$$

28 of $SO(8)$ decomposes as

$$\mathbf{28} = \mathbf{15}_0 + \mathbf{6}_2 + \bar{\mathbf{6}}_{-2} + \mathbf{1}_0. \quad (108)$$

Therefore, matter fields **6** (could) arise on the bulk A . The generations of **6** on the bulk A is given by:

$$n_{\mathbf{6}} - n_{\bar{\mathbf{6}}} = - \int_A c_1(A) c_1(\mathcal{L}^2) = -4(a+b). \quad (109)$$

$A \cap B_i = \Sigma_9$, $i = 1, 2$, and therefore, the bulk A contains 2 matter curves Σ_9 of genus 9. **8** of $SO(8)$ decomposes under (107) as

$$\mathbf{8} = \mathbf{4}_1 + \bar{\mathbf{4}}_{-1}. \quad (110)$$

Therefore, the matter fields **4** (could) localize along matter curves Σ_9 . Since f and g are bidegree (4,4) polynomials, the restriction \mathcal{L}_{Σ_9} of the line bundle \mathcal{L} to the matter curve Σ_9 has degree $4(a+b)$. The degree of the canonical bundle K_{Σ_9} is $2g - 2 = 16$. Let W be the divisor associated with the line bundle $K_{\Sigma_9}^{1/2} \otimes \mathcal{L}_{\Sigma_9}$, so that $\mathcal{O}_{\Sigma_9}(W) = K_{\Sigma_9}^{1/2} \otimes \mathcal{L}_{\Sigma_9}$. The degree of W is $8 + 4(a+b)$. Now, by the Riemann–Roch theorem,

$$\begin{aligned} n_{\mathbf{4}} - n_{\bar{\mathbf{4}}} &= h^0(W) - h^0(K_{\Sigma_9} - W) \\ &= \deg W + 1 - 9 \\ &= 4(a+b). \end{aligned} \quad (111)$$

Therefore, we have

$$n_{\mathbf{6}} - n_{\bar{\mathbf{6}}} = -(n_{\mathbf{4}} - n_{\bar{\mathbf{4}}}). \quad (112)$$

When $a+b > 0$, matter fields $\bar{\mathbf{6}}$ arise on the bulk A , and matter fields **4** localize along matter curves Σ_9 . Yukawa coupling that arises is

$$\bar{\mathbf{6}} \cdot \mathbf{4} \cdot \mathbf{4}. \quad (113)$$

When $a+b < 0$, matters **6** arise on the bulk A , and matter fields $\bar{\mathbf{4}}$ localise along matter curves Σ_9 . Yukawa coupling for this case is

$$\mathbf{6} \cdot \bar{\mathbf{4}} \cdot \bar{\mathbf{4}}. \quad (114)$$

Next, we consider the case in which component A_1 is coincident with two other components. Then, singular fiber on A_1 are enhanced to type III^* , and E_7 gauge group arises on the 7-branes wrapped on A_1 . We again abbreviate component A_1 to A . When E_7 breaks to E_6 under

$$E_7 \supset E_6 \times U(1), \quad (115)$$

133 of E_7 decomposes as

$$\mathbf{133} = \mathbf{78}_0 + \mathbf{27}_{-1} + \bar{\mathbf{27}}_1 + \mathbf{1}_0. \quad (116)$$

Therefore, matter fields **27** (could) arise on component A . The generations of **27** on the bulk A is given by:

$$n_{\mathbf{27}} - n_{\overline{\mathbf{27}}} = - \int_A c_1(A) c_1(\mathcal{L}^{-1}) = 2(a+b). \quad (117)$$

$A \cap B_i = \Sigma_9$, $i = 1, 2$, and therefore, the bulk A contains two matter curves Σ_9 of genus 9. **56** of E_7 decomposes under (115) as

$$\mathbf{56} = \mathbf{27}_1 + \overline{\mathbf{27}}_{-1} + \mathbf{1}_3 + \mathbf{1}_{-3}. \quad (118)$$

Therefore, matter fields **27** localize along the matter curves Σ_9 . The restriction \mathcal{L}_{Σ_9} of the line bundle \mathcal{L} to matter curve Σ_9 has degree $4(a+b)$. Let W be the divisor associated with the line bundle $K_{\Sigma_9}^{1/2} \otimes \mathcal{L}_{\Sigma_9}$, so that $\mathcal{O}_{\Sigma_9}(W) = K_{\Sigma_9}^{1/2} \otimes \mathcal{L}_{\Sigma_9}$. By applying the Riemann–Roch theorem, we find that the generation of **27** along matter curve Σ_9 is given by:

$$\begin{aligned} n_{\mathbf{27}} - n_{\overline{\mathbf{27}}} &= h^0(W) - h^0(K_{\Sigma_9} - W) \\ &= 4(a+b). \end{aligned} \quad (119)$$

When $a+b > 0$, matter fields **27** arise on the bulk A , and along matter curves Σ_9 . Yukawa coupling that arises is

$$\mathbf{27} \cdot \mathbf{27} \cdot \mathbf{27}. \quad (120)$$

When $a+b < 0$, matter fields $\overline{\mathbf{27}}$ arise on the bulk A , and along matter curves Σ_9 . Yukawa coupling for this case is

$$\overline{\mathbf{27}} \cdot \overline{\mathbf{27}} \cdot \overline{\mathbf{27}}. \quad (121)$$

There are only two matter curves Σ_9 , $A \cap B_1$ and $A \cap B_2$, in component A ; triple intersection of matter curves in bulk A does not occur for double covers of (18).

The results are shown in Table 11 below.

Gauge Group	$a+b$	Matter on E	# Gen. on E	Matter on Σ_1	# Gen. on Σ_1	Yukawa
E_7	> 0	27	$2(a+b)$	27	$4(a+b)$	27 · 27 · 27
	< 0	$\overline{\mathbf{27}}$	$-2(a+b)$	$\overline{\mathbf{27}}$	$-4(a+b)$	$\overline{\mathbf{27}} \cdot \overline{\mathbf{27}} \cdot \overline{\mathbf{27}}$
$SO(8)$	> 0	$\overline{\mathbf{6}}$	$4(a+b)$	4	$4(a+b)$	$\overline{\mathbf{6}} \cdot \mathbf{4} \cdot \mathbf{4}$
	< 0	6	$-4(a+b)$	$\overline{\mathbf{4}}$	$-4(a+b)$	$\mathbf{6} \cdot \overline{\mathbf{4}} \cdot \overline{\mathbf{4}}$

Table 11: Potential matter spectra for double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

6 Conclusions

We considered $(3,2,2,2)$ hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, and double covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ ramified over a $(4,4,4,4)$ 3-fold, to construct genus-one fibered Calabi–Yau 4-folds. By considering specific types of equations, we constructed two families of $(3,2,2,2)$ hypersurfaces, namely Fermat-type hypersurfaces and hypersurfaces in Hesse form. For double covers, we considered a family described by specific types of equations. We showed that these three families of genus-one fibered Calabi–Yau 4-folds lack a global section. Genus-one fibers of these families possess complex multiplication of specific orders, 3 and 4, and these symmetries enabled a detailed study of the gauge theories in F-theory compactifications.

We determined the discriminant loci of these families, and we specified the forms of the discriminant components and their intersections. In particular, discriminant components contain matter curves.

$SU(3)$ gauge groups generically arise on 7-branes wrapped on discriminant components in F-theory compactifications on Fermat-type $(3,2,2,2)$ hypersurfaces; when 7-branes coincide, the gauge symmetry is enhanced to E_6 . Only gauge groups of the form $SU(N)$ arise on 7-branes in F-theory compactifications on $(3,2,2,2)$ hypersurfaces in Hesse form; $SO(m)$ and E_n gauge groups do not form on 7-branes. $SU(2)$ gauge groups generically arise on 7-branes in F-theory compactifications on double covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. When 7-branes coincide, the $SU(2)$ gauge group is enhanced to $SO(8)$; when more 7-branes coincide, gauge group is enhanced to E_7 .

We specified the Mordell–Weil groups of Jacobian fibrations of specific Fermat-type hypersurfaces and specific double covers. They are \mathbb{Z}_3 and \mathbb{Z}_2 , such the Mordell–Weil groups have the rank 0, and F-theory compactifications on these specific Calabi–Yau genus-one fibrations do not have $U(1)$ gauge symmetry.

We computed the potential matter spectra and potential Yukawa couplings on specific components. We did not discuss the existence of a consistent four-form flux in this note. We showed that intrinsic 2-cycles are candidates for consistent four-form fluxes. We computed the Euler characteristics of Calabi–Yau 4-folds constructed in this note, in order to derive the conditions imposed on four-form fluxes to cancel the tadpole.

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